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The exact renormalisation group equation and the perturbed unitary minimal models

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Abstract: The exact renormalisation group equation is studied for a two-dimensional theory with exponential interaction and a background charge at infinity. The motivation for studying this interaction is the flow between unitary minimal models perturbed by $\Phi_{(1,3)}$, and their realisation in terms of a quantum group restricted sine-Gordon model.

Keywords: Renormalization Regularization and Renormalons, Field Theories in Lower Dimensions.

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Contents

1. Introduction

To get a better understanding of the space of quantum field theories, it is important to study the renormalisation group flow that determines its structure. Two-dimensional quantum field theories serve as a good starting point for two reasons. Firstly, the fixed points of the renormalisation group are two-dimensional conformal field theories which are well understood due to their large symmetry, and secondly, Zamolodchikov's c-theorem states that for a unitary theory the renormalisation group flows are irreversible flows in the coupling constant space.

The exact renormalisation group equation is a functional differential equation which describes how the wilsonian effective action must vary when the cut-off changes so that the physics is invariant. It is derived by integrating out the ultraviolet degrees of freedom in the partition function. In this paper the exact renormalisation group equation is studied for an exponential interaction and a background charge at infinity.

The motivation for considering this case is that the renormalisation group is known to flow between certain unitary conformal field theories, namely the unitary minimal models. A two-dimensional conformal field theory is specified by its field content, the operator product coefficients and the value of the Virasoro central charge c that measures the number of massless degrees of freedom in the theory. The unitary minimal models $\mathcal{M}_m$ ($m = 3, 4, \ldots$) are unitary conformal field theories with a finite number of primary fields (the highest weight vectors of the Virasoro algebra) and a central charge $0 < c = 1 - 6/m(m + 1) < 1$.

The unitary minimal models $\mathcal{M}_m$ have the same $(m - 1)$ critical behaviour as Landau-Ginzburg theory with a bosonic field and even polynomial interactions up to the power $2(m-1)$ [1]. The minimal model $\mathcal{M}_m$ perturbed by the relevant field $\Phi_{(1,3)}$ was in [2] argued to have a renormalisation group flow from $\mathcal{M}_m$ in the ultraviolet to

$\mathcal{M}_{m-1}$ in the infrared limit using the thermodynamic Bethe ansatz. Indeed, it was shown perturbatively in the double scaling limit of small coupling g and large m that the infrared central charge $c_{\rm IR}$ satisfies that $c_{\rm IR} = c_{\mathcal{M}_{m-1}} + O(m^{-4})$. This calculation was performed to first order in [3, 4] and second order in [5]. The ultraviolet fixed point is at $g_{UV} = 0$ and in [3, 4] it was shown that the infrared fixed point is at $g_{\rm IR} \propto \frac{4}{m+1}$ so that perturbation theory is a good approximation for the infrared theory in the limit $m \to \infty$ where $g_{\rm IR} \to 0$.

The exact renormalisation group equation is not solvable and has to be approximated, we will do this by including only an exponential interaction together with a background charge in the wilsonian effective action. Using the equivalence between the perturbed minimal models and the quantum group restricted sine-Gordon model, we will argue that this interaction describes the renormalisation group flow of the unitary minimal models perturbed by $\Phi_{(1,3)}$.

In section 2 the unitary minimal model perturbed by $\Phi_{(1,3)}$ is described together with its realisation as a quantum group restricted sine-Gordon model. In section 3 the exact renormalisation group equation is introduced, and it is approximated by only allowing relevant operators in the effective action, which for the minimal models are given by exponential operators. The wilsonian effective potential does not contain any field derivatives so the approximation is similar to the local potential approximation. The non-linear term in the exact renormalisation group equation is approximated using the operator product expansion.

For the perturbed minimal models a renormalisation group equation is obtained where all higher-order terms in the coupling are contained in the off-critical structure constant for the operator product expansion. This renormalisation group equation is valid for all $m > 3$, and in the perturbative limit $m \to \infty$ it is equal to the perturbative renormalisation group equation.

2. Unitary minimal models perturbed by $\Phi_{(1,3)}$

We denote by $\mathcal{M}_m$ the unitary minimal model that has the spin zero primary fields $\Phi_{(p,q)}$, $1 \le p \le m-1$, $1 \le q \le m$, and a central charge $c = 1 - 6/m(m+1)$. $\Phi_{(p,q)}$ has the conformal scaling dimension $h_{(p,q)} = \bar{h}_{(p,q)} = \frac{((m+1)p-mq)^2-1}{4m(m+1)}$ which shows that $\Phi_{(p,q)} = \Phi_{(m-p,m+1-q)}$ leaving only $m(m-1)/2$ different primary fields. The unitary minimal model $\mathcal{M}_m$ perturbed by the relevant operator $\Phi_{(1,3)}$ can formally be written as[1]

$$S = \mathcal{M}_m + \lambda \int d^2z \; \Phi_{(1,3)}(z,\bar{z}) \,, \tag{2.1}$$

[1]The standard conventions are used (see [3]), with the normalisation $d^2z \equiv 2idz\wedge d\bar{z} = dx\wedge dt_E = d^2x$, and positive structure constant $C^{(1,3)}_{(1,3)(1,3)} = C^{\Phi}_{\Phi\Phi} > 0$.

and the theory is defined by the correlators

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \frac{\left\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{-\lambda\int d^2z\Phi_{(1,3)}}\right\rangle_{\mathcal{M}_m}}{\left\langle e^{-\lambda\int d^2z\Phi_{(1,3)}}\right\rangle_{\mathcal{M}_m}}, \tag{2.2}$$

where $\langle\cdots\rangle_{\mathcal{M}_m}$ is the correlator in the minimal model $\mathcal{M}_m$ and $\mathcal{O}_i(x)$ are the local scaling fields in the theory. The renormalisation group eigenvalue for $\Phi_{(1,3)}$ follows from (2.1) using dimensional analysis: $y = 2 - \Delta = \frac{4}{m+1}$ where $\Delta = 2h_{(1,3)}$. $\Phi_{(1,3)}$ is therefore a relevant operator ($y > 0$), and from the form of $h_{(p,q)}$ above it directly follows that the other relevant primary fields are $\Phi_{(p,p+s)}$ with $-1 \le s \le 2$ and $0 < p+s < m+1$ for $1 \le p \le m-1$. There are therefore $2m-3$ different relevant primary fields including $\Phi_{(1,1)} = \mathbf{1}$. These are the only relevant scalar fields as all the spin zero descendants will have scaling dimension $2h_{(p,q)} + 2n$ with $n \in \mathbb{N}$.

In the Coulomb gas [6] (or Feigin-Fuchs, Dotsenko-Fateev) representation the minimal model $\mathcal{M}_m$ is realised as a lagrangian free field theory with a background charge $-2\alpha_0$ at infinity and a central charge $c = 1 - 24\alpha_0^2$. $\alpha_0 > 0$ so that $\alpha_0 = 1/\sqrt{4m(m+1)}$. The action is then

$$S = \frac{1}{8\pi}\int d^2x\sqrt{g(x)}\left(g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + 2\sqrt{2}i\alpha_0\phi(x)R(x)\right) \tag{2.3}$$

on the Riemann sphere $\mathbb{C}\cup\{\infty\}$. The primary fields are the vertex operators $V_{\alpha_{r,s}}(x) = e^{i\sqrt{2}\alpha_{r,s}\phi(x)}$ with $\alpha_{r,s} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_-$ where $\alpha_-\alpha_+ = -1$ and $\alpha_- + \alpha_+ = 2\alpha_0$ so that $\alpha_- = -\sqrt{\frac{m}{m+1}}$. The vertex operator conformal dimension becomes

$$h_{r,s} = \bar{h}_{r,s} = \alpha_{r,s}(\alpha_{r,s} - 2\alpha_0) \tag{2.4}$$

and therefore the primary field $\Phi_{(1,3)}$ is represented as $e^{-i\sqrt{2}\alpha_-\phi}$. Note that $h_{\alpha_\pm} = \alpha_\pm(\alpha_\pm - 2\alpha_0) = 1$. To calculate correlators of vertex operators screening charges $Q_\pm = \int d^2zV_{\alpha_\pm}$ are added so that the overall charge of the correlator vanishes [6]. The presence of a background charge changes the energy-momentum tensor

$$T = -\frac{1}{2}\partial\phi\partial\phi \xrightarrow{-2\alpha_0} T = -\frac{1}{2}\partial\phi\partial\phi + i\sqrt{2}\alpha_0\partial^2\phi\,, \tag{2.5}$$

and therefore also the scaling behaviour of the theory lowering the central charge from the value of the free theory $c = 1$ to $c = 1 - 24\alpha_0^2$.

In [7, 8, 9] it was observed that the perturbed unitary minimal model (2.1) is equivalent to a quantum group restriction of the sine-Gordon theory. The sine-Gordon theory is given by the euclidean action

$$S_{SG} = \frac{1}{4\pi}\int d^2x\left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - 4\pi\tilde{\lambda}\cos\left(\beta\frac{\phi(x)}{\sqrt{4\pi}}\right)\right). \tag{2.6}$$

In two dimensions all polynomial interactions are super-renormalisable and all divergences appear in the tadpole diagrams which are the self contractions of a vertex. The renormalisation can be done by normal ordering which removes all self contractions, and this corresponds to a renormalisation of $\tilde{\lambda}$, but β and ϕ are not renormalised [10].

In [7] the quantum group restriction of the sine-Gordon model is performed by adding a background charge $-2\alpha_0$ at infinity. The background charge will again change the energy-momentum tensor as in (2.5) and the two vertex operators in the potential will now scale differently. Two different couplings are needed, we will write this as

$$\tilde{\lambda}\cos\left(\beta\frac{\phi}{\sqrt{4\pi}}\right) \xrightarrow{-2\alpha_0} \frac{\tilde{\lambda}}{2}e^{-i\beta\phi/\sqrt{4\pi}} + \frac{\tilde{\lambda}_-}{2}e^{i\beta\phi/\sqrt{4\pi}}\,. \tag{2.7}$$

Analogous to the Coulomb gas representation for the minimal models the background charge α_0 is determined from the central charge $c = 1-24\alpha_0^2 = 1-\frac{6}{m(m+1)}$. β is chosen from the requirement that $e^{i\beta\phi/\sqrt{4\pi}}$ becomes marginal (i.e. $h=\bar{h}=1$) so that it survives in the ultraviolet limit. $\beta = \sqrt{8\pi}\alpha_- = -\sqrt{8\pi}\sqrt{m/(m+1)}$. The conformal dimension of $e^{-i\beta\phi/\sqrt{4\pi}}$ is then $h = \frac{m}{m+1}-\frac{1}{m+1} = \frac{m-1}{m+1} = h_{(1,3)}$ and $e^{-i\beta\phi/\sqrt{4\pi}}$ represents the perturbing operator $\Phi_{(1,3)}$.[2] Only relevant operators are included in the effective action, as explained below, so the marginal screening term $e^{i\beta\phi/\sqrt{4\pi}}$ (which is needed when considering correlators) is excluded.

In [7] it was argued that the perturbed unitary minimal model (2.1) is a quantum group restriction of the sine-Gordon theory, i.e. a massive theory, with the coupling taken as $\tilde{\lambda} = -\lambda$ in (2.6). Here we are only interrested in the massless flow to $\mathcal{M}_{m-1}$ ($\lambda > 0$ in [3]) where the limit $m\to\infty$ is given by perturbation theory. We will therefore take the opposite sign, $\tilde{\lambda} = k\lambda$ for some $k\in\mathbb{R}_+$, and in the exact renormalisation group equation we consider the action

$$S = \frac{1}{4\pi}\int d^2x\left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - 4\pi\frac{\tilde{\lambda}}{2}e^{-i\beta\phi(x)/\sqrt{4\pi}} + \sqrt{2}i\alpha_0\phi(x)R(x)\right) \tag{2.8}$$

as a model for the perturbed unitary minimal model (2.1). The operator $\Phi_{(1,3)}$ in (2.1) is normalised as in [12] so that the ultraviolet limit of correlators in the perturbed theory (2.2) equals the corresponding correlators in $\mathcal{M}_m$ where we take $\langle\Phi_{(1,3)}(1)\Phi_{(1,3)}(0)\rangle_{\mathcal{M}_m} = 1$. The scaling dimension of $V_{(1,3)}$ and $\Phi_{(1,3)}$ are equal[3] so the normalisation constant relating them is independent of $\tilde{\lambda}$, and is given by the formulas in [6][4] $V_{(1,3)} = N_{(1,3)}\Phi_{(1,3)}$, we therefore take $\tilde{\lambda} = {N_{(1,3)}}^{-1}\lambda$.

[2]The ultraviolet limit of the restricted sine-Gordon model is a Liouville theory which is equivalent to the unitary minimal models $\mathcal{M}_m$, the perturbed minimal models can therefore also be seen as a perturbed Liouville theory [11].

[3]In [27] the exact relation between the sine-Gordon coupling $\tilde{\lambda}$ (i.e. without a background charge, so that the scaling dimensions of $\cos(\beta\phi/\sqrt{4\pi})$ and $\Phi_{(1,3)}$ here differ) and λ was established $\lambda = \frac{\tilde{\lambda}^2\pi}{4(1-\beta^2/(4\pi))(3\beta^2/(8\pi))}(\gamma(1-\beta^2/(8\pi))^3\gamma(3\beta^2/(8\pi)))^{1/2}$ using the Coulomb gas representation of minimal models [6] ($\gamma(x) = \Gamma(x)/\Gamma(1-x)$). We do not use this representation here because we want the ultraviolet limit to be $\mathcal{M}_m$, and not as in the sine-Gordon model where it is the free theory with $c=1$, also we want a linear relation between λ and $\tilde{\lambda}$.

[4]${N_{(1,3)}}^2 = 2\pi^2\frac{\gamma(1-\alpha_-^2)^3\gamma(3\alpha_-^2-1)}{(1-2\alpha_-^2)(2\alpha_-^2-1)}$ with $N_{(1,3)} > 0$.

For a non-zero background charge the scaling of the theory is not determined by the usual β-function related to the ultraviolet divergences, but by a generalised β-function as discussed in [13]. The generalised β-function takes into account the change in the energy-momentum tensor when the background charge is added. To lowest order the background charge adds a term [13]

$$\beta(\tilde{\lambda}) = \Lambda \frac{\partial \tilde{\lambda}}{\partial \Lambda} \longrightarrow \tilde{\beta}(\tilde{\lambda}) = \beta(\tilde{\lambda}) - 2 \cdot 2\alpha_{(1,3)}\alpha_0 \tilde{\lambda} \tag{2.9}$$

to the normal perturbative β-function for the interaction term $\tilde{\lambda} e^{i\sqrt{2}\alpha_{(1,3)}\phi(x)}$ in the action. This follows from the change in the scaling dimension of $\Phi_{(1,3)}$: $\Delta \to \Delta - 2 \cdot 2\alpha_{(1,3)}\alpha_0$ with a background charge (2.4).[5] We discuss below how this term appears in the exact renormalisation group.

3. The exact renormalisation group

The wilsonian effective action at the scale Λ is obtained by "integrating out" momentum modes between Λ and Λ_0, where Λ_0 is the fundamental wilsonian cut-off. The exact renormalisation group equation[6] then describes how the effective action must change so as to describe the same physics when Λ changes. Following [14], the partition function is written as

$$Z(\Lambda) = \int D\phi \; e^{-S[\phi]} = \int D\phi \; e^{-\frac{1}{2}\int \frac{d^2p}{(2\pi)^2} \phi(p)\phi(-p) p^2 K^{-1}(p^2/\Lambda^2) - V(\phi,\Lambda)} \tag{3.1}$$

in a continuum formulation with the cut-off propagator $\frac{K(p^2/\Lambda^2)}{p^2}$. $K(p^2/\Lambda^2)$ is constant for small $s = p^2/\Lambda^2$ and vanishes faster than any power for large s. The physics must be independent in the choice of effective scale Λ so that $\Lambda \frac{d}{d\Lambda} Z(\Lambda) = 0$, and from (3.1) it follows that

$$\begin{aligned} \Lambda \frac{d}{d\Lambda} Z(\Lambda) = \Bigg\langle \bigg(&-\frac{1}{2} \int \frac{d^2p}{(2\pi)^2} \phi(p)\phi(-p) p^2 \Lambda \frac{\partial K^{-1}(p^2/\Lambda^2)}{\partial \Lambda} - \\ &- \Lambda \frac{\partial V(\phi,\Lambda)}{\partial \Lambda} \bigg) \Bigg\rangle = 0\,. \end{aligned} \tag{3.2}$$

This condition holds when the wilsonian effective potential satisfies the operator equation[7]

$$\begin{aligned} \Lambda \frac{\partial}{\partial \Lambda} V(\phi,\Lambda) = \frac{1}{2} \int d^dx d^dy \int \frac{d^dk}{(2\pi)^d} e^{ik(x-y)} \frac{2}{\Lambda^2} K'\left(k^2/\Lambda^2\right) \times \\ \times \left(\frac{\delta^2 V}{\delta\phi(x)\delta\phi(y)} - \frac{\delta V}{\delta\phi(x)} \frac{\delta V}{\delta\phi(y)} \right), \end{aligned} \tag{3.3}$$

[5] The β-function starts as $-y\tilde{\lambda}$ where y is the renormalisation group eigenvalue.
[6] Or the Wilson-Polchinski renormalisation group equation [15, 14].
[7] A constructive proof is given in [16] by explicit functional integration.

because the partition function then changes as a total derivative (up to a field independent term) $\Lambda\frac{d}{d\Lambda}Z = \int d^2p \int D\phi \frac{\delta}{\delta\phi}\Psi = 0$ [14]. $K'(k^2/\Lambda^2) = \frac{dK(s)}{ds}|_{s=k^2/\Lambda^2}$. (3.3) is the exact renormalisation group equation. It is a non-linear functional equation in $V(\phi,\Lambda)$ which is not directly solvable and approximations must be made, either by truncating the operator space or by performing a derivative expansion [17].

We will make a truncation in the operator space. Firstly, we will only consider relevant operators, this approximation becomes exact in the infrared ($\Lambda \ll \Lambda_0$) where the effective theory is determined by the relevant (and marginal) couplings. Secondly, it is known that the primary fields $\{\Phi_{(1,2p+1)}\}$ with $0 \le p \le [\frac{m-1}{2}]$ form a sub-algebra[8] in $\mathcal{M}_m$, and the renormalisation group flow from the minimal model $\mathcal{M}_m$ is therefore spanned by these operators.[9]

In the sub-algebra $\{\Phi_{(1,2p+1)}\}$ only $\Phi_{(1,1)} = \mathbf{1}$ and $\Phi_{(1,3)}$ are relevant fields.[10] The exact renormalisation group is only determined up to a field independent term, so we will only consider perturbations with respect to $\Phi_{(1,3)}$.[11] None of the relevant operators $V_{(1,n)}$ contain any derivatives in ϕ, so the situation is similar to the local potential approximation of the exact renormalisation group equation. The local potential approximation is the lowest order term in a derivative expansion of $V = V(\Lambda,\phi,\partial\phi,\ldots)$ [20]. The second order term containing derivatives $f(\Lambda)(\partial\phi)^2$ leads to a renormalisation of ϕ and an anomalous dimension $\eta > 0$. In our approximation there are no derivatives in ϕ because we only keep relevant operators, and ϕ is not renormalised in the sine-Gordon model. In this approximation the first term in (3.3) can be rewritten [20] as

$$\frac{1}{\Lambda^2}\int \frac{d^2k}{(2\pi)^2} K'\left(k^2/\Lambda^2\right) \int d^2x \frac{\partial^2 \mathcal{V}}{\partial\phi\partial\phi}(x) = -\gamma_1 \int d^2x \frac{\partial^2 \mathcal{V}}{\partial\phi\partial\phi}(x)\,, \tag{3.4}$$

where $\mathcal{V}(x)$ is the potential density $V = \int d^2x \mathcal{V}(x)$, and $\gamma_1 = -\int \frac{d^2k}{(2\pi)^2}\ K'(k^2) > 0$ as $K(k^2)$ is decreasing. In the local potential approximation [21, 20, 22] the second non-local term in (3.3)

$$-\int d^2x \int d^2y \int \frac{d^2k}{(2\pi)^2} K'\left(k^2/\Lambda^2\right) e^{ik(x-y)} \frac{\partial \mathcal{V}}{\partial\phi(x)} \frac{\partial \mathcal{V}}{\partial\phi(y)} \tag{3.5}$$

is approximated by $\gamma_2 \int d^2x (\frac{\partial\mathcal{V}}{\partial\phi(x)})^2$ where $\gamma_2 \in \mathbb{R}_+$ depends on the cut-off function $K(s)$. This is a good approximation in the ultraviolet limit $\Lambda \to \infty$ [22] where $\int \frac{d^2k}{(2\pi)^2} K'(k^2/\Lambda^2) e^{ik(x-y)} \to K'(0) \int \frac{d^2k}{(2\pi)^2} e^{ik(x-y)} = K'(0)\delta^2(x-y)$. We consider the

[8] $\Phi_{(1,r)}\Phi_{(1,s)} = \sum_p C^p_{rs}\Phi_{(1,p)}$ where $p \in \{|r-s|+1, |r-s|+3, \ldots, r+s-1\}$ in steps of two [6].

[9] The operator structure at the non-trivial ultraviolet fixed point thereby determines the renormalisation group flow. When considering perturbations from operators in a sub-algebra, all divergences that arise away from the fixed point will be contained in this sub-algebra showing that there is a renormalisation group flow in the sub-space of the corresponding couplings.

[10] The operator $\Phi_{(1,3)}$ does not mix with other relevant operators off criticality [3, 18].

[11] The integrable perturbation with $\Phi_{(1,3)}$ was first considered in [19].

operator version of (3.3) and a different approximation therefore has to be used for the term (3.5) otherwise divergences will appear in the operator product $\frac{\partial \mathcal{V}}{\partial\phi(x)}\frac{\partial \mathcal{V}}{\partial\phi(x)}$, whereas (3.3) is finite for $\Lambda < \infty$. The correct form of the approximation in the operator case is obtained from the operator product expansion. For the spin zero operators is [23]

$$\mathcal{O}_i(x)\mathcal{O}_j(y) \sim \sum_k |x|^{\Delta_k-\Delta_i-\Delta_j}\tilde{C}_{ij}^k(\tilde{\lambda},\Lambda)\mathcal{O}_k(y)\,, \tag{3.6}$$

where $\mathcal{O}_k(y)$ is a complete set of local scaling fields. The non-local term (3.5) then takes the form

$$\int d^2x\mathcal{O}(x)\int d^2y\int \frac{d^2k}{(2\pi)^2}e^{ik(x-y)}K'\left(k^2/\Lambda^2\right)h(|x-y|^2) \tag{3.7}$$

for some $h(|x-y|^2)$ and operator $\mathcal{O}(x)$. We use that $\delta^2(x) = \frac{1}{\pi}\delta(|x|^2)$ to write the ultraviolet limit $\Lambda\to\infty$ of the k integral as $-\int \frac{d^2k}{(2\pi)^2}e^{ik(x-y)}K'(k^2/\Lambda^2) \to \gamma_2\delta(|x-y|^2)$ where again $\gamma_2 \in \mathbb{R}_+$ depends on the choice of $K(s)$. For finite Λ this expression is approximately $\gamma_2\delta(|x-y|^2 - a/\Lambda^2)$ where $a\in\mathbb{R}_+$ depends on $K(s)$. The cut-off dependence in a is removed by redefining $\Lambda^2 \to a\Lambda^2$, and the approximation becomes

$$-\int \frac{d^2k}{(2\pi)^2}e^{ik(x-y)}K'\left(k^2/\Lambda^2\right) = \gamma_2\;\delta\left(|x-y|^2-\Lambda^{-2}\right) \tag{3.8}$$

in (3.5).[12] If the effective potential is written as $V = \tilde{\lambda}^i\int d^2x\mathcal{O}_i(x)$, where $[\tilde{\lambda}^i] = y^i > 0$, then (3.3) becomes using (3.4) and the operator product expansion

$$\begin{aligned}\frac{\gamma_1}{\gamma_2}\int d^2x\Lambda\frac{d}{d\Lambda}\mathcal{V}(\phi,\Lambda) = &-\frac{\gamma_1}{\gamma_2}\int d^2x\frac{\partial^2\mathcal{V}(\phi,\Lambda)}{\partial\phi\partial\phi} + \\ &+\frac{\gamma_1}{\gamma_2}\frac{\pi}{\Lambda^2}\int d^2x\sum_{i,j,k}\tilde{\lambda}^i\tilde{\lambda}^j\Lambda^{\Delta'_i+\Delta'_j-\Delta'_k}\tilde{C}'^k_{ij}(\tilde{\lambda},\Lambda)\mathcal{O}'_k(x)\,.\end{aligned} \tag{3.9}$$

Here V and ϕ have been rescaled: $V\to\frac{\gamma_1}{\gamma_2}V$, $\phi\to\sqrt{\gamma_1}\phi$ and Δ'_i and $\tilde{C}'^k_{ij}$ are the scaling dimension and structure constant for the field $\mathcal{O}'_i = \frac{\partial\mathcal{O}_i}{\partial\phi}$. If $\mathcal{O}_i$ is a vertex operator then $\Delta' = \Delta$, $\tilde{C}'^k_{ij} \propto \tilde{C}^k_{ij}$ and $\mathcal{O}'_i \propto \mathcal{O}_i$. It follows from (3.9) that the implicit dependence in the choice of cut-off function $K(s)$ contained in γ_1 and γ_2 drops out [20]. The renormalisation group equation is usually written in terms of dimensionless couplings $\tilde{g}^i = \Lambda^{-y_i}\tilde{\lambda}^i$ so that $V = \tilde{g}^i\Lambda^2\int d^2x\tilde{\mathcal{O}}_i$ where $[\tilde{\mathcal{O}}_i] = 0$. (3.9) then becomes

$$\begin{aligned}\Lambda^2\int d^2x\Lambda\frac{\partial\tilde{g}^i(\Lambda)}{\partial\Lambda}\tilde{\mathcal{O}}_i = &-2\Lambda^2\int d^2x\tilde{g}^i\tilde{\mathcal{O}}_i - \Lambda^2\int d^2x\tilde{g}^i\frac{\partial\tilde{\mathcal{O}}_i}{\partial\phi\partial\phi} + \\ &+\pi\Lambda^2\int d^2x\sum_{i,j,k}\tilde{g}^i\tilde{g}^j\tilde{C}'^k_{ij}\tilde{\mathcal{O}}'_k\,.\end{aligned} \tag{3.10}$$

[12] Adding a function $f(|x-y|^2, 1/\Lambda^2)$ satisfying $f\to 0$ for $\Lambda\to\infty$ corresponds to a higher-order derivative expansion in ϕ [22] and it is therefore neglected in the approximation considered here where there are no derivatives in ϕ.

For the perturbed minimal models we will use the action (2.8), and for the moment neglect the curvature term. (3.10) then becomes

$$\int d^2x e^{-i\beta\phi/\sqrt{4\pi}} \left(\Lambda \frac{\partial \tilde{g}}{\partial \Lambda} + 2\tilde{g}(\Lambda) + \tilde{g}(\Lambda) \left(\frac{-i\beta}{\sqrt{4\pi}} \right)^2 + \right.$$
$$\left. + \pi \frac{\tilde{g}(\Lambda)^2}{2} \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(\Lambda), \Lambda) \left(\frac{-i\beta}{\sqrt{4\pi}} \right)^2 \right) = 0 \,,$$

up to field independent and irrelevant terms. Introducing the dimensionless variable $t = \ln \frac{\Lambda_0}{\Lambda}$ (and $\tilde{x} = \Lambda x$) the renormalisation group equation for $\tilde{g}$ can then be written as

$$\frac{\partial \tilde{g}(t)}{\partial t} = \dot{\tilde{g}} = 2\tilde{g}(t) - \frac{2m}{m+1}\tilde{g}(t) - \pi \frac{m}{m+1} \tilde{g}(t)^2 \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t) \,. \tag{3.11}$$

Adding the additional term (2.9) from the background charge $2 \cdot 2\alpha_{(1,3)}\alpha_0 \tilde{g}(t) = \frac{2}{m+1}\tilde{g}(t)$ gives

$$\begin{aligned} \dot{\tilde{g}}(t) &= \frac{4}{m+1}\tilde{g}(t) - \pi \frac{m}{m+1} \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)\tilde{g}(t)^2 \\ &= y\tilde{g}(t) - \pi \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)\tilde{g}(t)^2 \frac{m}{m+1} \,. \end{aligned} \tag{3.12}$$

The contribution from the background charge to the scaling behaviour can be incorporated directly into the exact renormalisation group equation if the curvature term in (2.8) is taken into account. The action (2.8) is defined on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ where all the curvature is situated at infinity so that the topological invariant $\int d^2\tilde{x} \sqrt{g(\tilde{x})} R(\tilde{x}) = 8\pi$ is satisfied. $\mathbb{C} \cup \{\infty\}$ cannot be covered by a single coordinate chart, one chart is needed for the flat space and another in the neighbourhood of infinity. We take polar coordinates in flat space $x = (r \sin\theta, r\cos\theta)$ and define the contribution from infinity as in [24] by a limit value. We consider the theory on a disk Γ with radius $r \to \infty$ so that all curvature is at the boundary $\partial\Gamma$. From the Gauss-Bonnet theorem $\lim_{r\to\infty} \int_{\partial\Gamma} d\theta \; rR(r) = 8\pi$ so that $R(r) = 4/r$ on $\partial\Gamma$. We define as in [24] the contribution from infinity to be $\phi_\infty = \lim_{r\to\infty} \frac{1}{2\pi r} \int_{\partial\Gamma} dl \; \phi = \lim_{r\to\infty} \frac{1}{2\pi} \int_{\partial\Gamma} d\theta \; \phi$ and similarly for the vertex operators. The effect of the background term is seen by evaluating the exact renormalisation group equation at infinity where the curvature is non vanishing. Hence we replace $\int d^2x$ in (3.10) by $\lim_{r\to\infty} \frac{1}{2\pi} \int_{\partial\Gamma} d\theta$. Only the term $\frac{\partial V}{\partial\phi}\frac{\partial V}{\partial\phi}$ in the exact renormalisation group equation (3.3) will give an additional term proportional to $e^{-i\beta\phi/\sqrt{4\pi}}$ when the background charge is added. This term becomes

$$\begin{aligned} &\lim_{r\to\infty} \frac{-1}{2\pi} \int_{\partial\Gamma} d\theta \int d^2y \delta\left(|x-y|^2 - \Lambda^{-2}\right) \frac{2^2(\sqrt{2})^2(i)^2\alpha_0\alpha_{(1,3)}}{8\pi} rR(r) e^{i\sqrt{2}\alpha_{(1,3)}\phi(y)} \tilde{g}(t) = \\ &= (2)^3 \alpha_0\alpha_{(1,3)}\tilde{g}(t) \lim_{r\to\infty} \int_{\partial\Gamma} \frac{d\theta}{2\pi} \int \frac{d\theta'}{2\pi} \int \frac{d(r')^2}{2} \delta\left((r')^2 - \Lambda^{-2}\right) e^{i\sqrt{2}\alpha_{(1,3)}\phi(x-y')} \\ &\approx 2^2\alpha_0\alpha_{(1,3)}\tilde{g}(t) \lim_{r\to\infty} \int_{\partial\Gamma} \frac{d\theta}{2\pi} e^{i\sqrt{2}\alpha_{(1,3)}\phi(x)} = 2^2\alpha_0\alpha_{(1,3)}\tilde{g}(t) V_{(1,3)}(\infty) \,. \end{aligned}$$

The second to last equation holds in the limit $r \to \infty$ where $r \gg \Lambda^{-1}$. When the coefficient of $e^{-i\beta\phi/\sqrt{4\pi}}$ from this term is added to (3.11) then (3.12) is again obtained, but it now holds to all orders in the coupling $\tilde{g}$.

This shows how the change in scaling behaviour due to the background charge is seen in the exact renormalisation group equation when it is evaluated at a point with non-zero curvature. Equation (3.12) is then valid to all orders in perturbation theory for the chosen truncation of the operator space. Hence, in

$$\dot{\tilde{g}}(t) = y\tilde{g}(t) - \pi \tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)\tilde{g}(t)^2 \frac{m}{m+1} \tag{3.13}$$

higher-order terms appear via the off-critical structure constant $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)$. The structure constant $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)$ is regular in the coupling $\tilde{g}$ [25, 26, 23]. The wilsonian effective action (2.8) is for zero coupling equal to the Coulomb gas representation (2.3) of the minimal model $\mathcal{M}_m$. The structure constant therefore has the following expansion $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t) = \tilde{C}^{\Phi}_{\Phi\Phi} + O(\tilde{g}(t))$, where $\tilde{C}^{\Phi}_{\Phi\Phi}$ is the structure constant for the vertex operators $V_{(1,3)}$ in the minimal model $\mathcal{M}_m$. $V_{(1,3)}(1)V_{(1,3)}(0) = \mathbf{1} + \tilde{C}^{\Phi}_{\Phi\Phi}V_{(1,3)}(0) + \cdots$ and the structure constant for $\Phi_{(1,3)}$ is thus $C^{\Phi}_{\Phi\Phi} = N^{-1}_{(1,3)}\tilde{C}^{\Phi}_{\Phi\Phi}$ [6].[13]

It follows from (3.13) by inserting $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t) = \tilde{C}^{\Phi}_{\Phi\Phi} + O(\tilde{g}(t))$ that the infrared fixed point coupling vanishes in the limit $m \to \infty$, and this limit can therefore be compared with the perturbative renormalisation group equation. For large m (3.13) becomes $\dot{g} = yg - \pi C^{\Phi}_{\Phi\Phi}g^2 + O(g^3)$ where $g = N_{(1,3)}\tilde{g}$, and this has the infrared fixed point $g_{\mathrm{IR}} = \frac{y}{\pi C^{\Phi}_{\Phi\Phi}}$ as obtained in [3, 4].

4. Conclusion

We studied the exact renormalisation group equation for a two-dimensional quantum field theory. It was approximated by only including relevant operators of exponential form together with a background charge at infinity in the wilsonian effective action. The effective action does not contain any derivatives in the field and the approximation is therefore similar to the local potential approximation, the non-linear term in

[13] $(C^{\Phi}_{\Phi\Phi})^2 = \frac{16}{3}\frac{(1-y)^4}{(1-y/2)^2(1-3y/4)^2}\left(\frac{\Gamma(1+y/2)}{\Gamma(1-y/2)}\right)^4\left(\frac{\Gamma(1-y/4)}{\Gamma(1+y/4)}\right)^3\left(\frac{\Gamma(1+y)}{\Gamma(1-y)}\right)^2\left(\frac{\Gamma(1+3y/4)}{\Gamma(1-3y/4)}\right)$, $C^{\Phi}_{\Phi\Phi} > 0$. $\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}(t), t)$ can in principle be calculated in conformal perturbation theory [23, 25, 26] for strictly relevant perturbations ($y > 0$), and for small t is it given by [23]

$$\tilde{C}^{\Phi}_{\Phi\Phi}(\tilde{g}) = \tilde{C}^{\Phi}_{\Phi\Phi} - \tilde{g}\pi \int' d^2z \langle V_{(1,3)}(\infty)V_{(1,3)}(z)V_{(1,3)}(1)V_{(1,3)}(0)\rangle_{\mathcal{M}_m} + \mathcal{O}(\tilde{g}^3)\,. \tag{3.14}$$

The four-point function is the first non-trivial correlator and it can be evaluated by pairing the fields and inserting the short distance expansion. Unfortunately no closed form has been found for the conformal blocks with two screening charges ($\langle VVVV\rangle$ requires the screening Q_-^2). When only one screening Q_- is needed a closed expression of the conformal blocks can be found in terms of hypergeometric functions using Euler's integral representation. In the marginal case $y = 0$ the four-point function can be written in a closed form as the conformal blocks become meromorphic functions of z [5], but conformal perturbation theory is only defined for strictly relevant perturbations with $y > 0$.

the exact renormalisation group equation is approximated by the operator product expansion. We showed that the effect of the background charge can be incorporated into the exact renormalisation group equation by evaluating it at a point of non-zero curvature.

Using the equivalence between the unitary minimal models perturbed by $\Phi_{(1,3)}$ and the quantum group restricted sine-Gordon model, the obtained renormalisation group equation was argued to describe the renormalisation group flow for the perturbed unitary minimal models from $\mathcal{M}_m$ to $\mathcal{M}_{m-1}$. The resulting renormalisation group equation is valid to all orders in the coupling for our truncation of the operator space (and for all $m > 3$). The higher-order terms in the coupling appear in the off-critical structure constant. In the limit of large m, where the ultraviolet and infrared fixed points approach each other, the renormalisation group equation agrees with the well-known perturbative result.

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